The Rational Sextic Curve, and the Cayley Symmetroid. * †

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§ 1.

The rational space sextic curve, ρ_3^6, may be regarded as determined by a single form involving binary and quaternary variables,

$$(a\xi)\,(\alpha t)^6 \equiv (m\,t)^6 = 0, \tag{1}$$

where the symbolic a's and α's must be taken in combination to have an actual meaning. Evidently (1) is the equation in plane coördinates of the point of the curve to which is attached the parameter t. Fixing ξ, (1) gives the parameters of the six points in which the plane ξ meets ρ_3^6. The notation $(m\,t)^6 = 0$, where m is a form whose coefficients are linear homogeneous functions of the four ξ's, will often be convenient.

Any invariant of the form $(m\,t)^6$ equated to zero represents a contravariant surface of (1); it is precisely the locus of planes which meet ρ_3^6 in six points whose parameters are the roots of a binary sextic for which the given invariant vanishes. An important invariant of the binary sextic is the so-called *catalecticant* — the invariant whose vanishing is the condition that the sextic have an apolar cubic. For the form (1) this is the symmetrical determinant

$$\begin{vmatrix} m_0, & m_1, & m_2, & m_3 \\ m_1, & m_2, & m_3, & m_4 \\ m_2, & m_3, & m_4, & m_5 \\ m_3, & m_4, & m_5, & m_6 \end{vmatrix} = 0. \tag{2}$$

The m's being linear in the coefficients ξ_i, (2) represents a surface of class 4. Cayley ‡ studied the quartic surface defined by a symmetrical determinant of order 4 whose elements are linear in the coördinates of a point of space. He called this surface the *symmetroid* and deduced many of its properties. It has ten nodes; these nodes are a symmetrical group of ten points such that any nine are projected from the tenth into the base-points of a pencil of plane cubics. The ten minors of the symmetrical determinant represent the ten linearly independent cubic surfaces on the ten nodes. (2) may be called a

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‡ Cayley, "A Memoir on Quartic Surfaces," *Proceedings of the London Mathematical Society*, Vol. III, pp. 19–69.

Cayley symmetroid; it is the dual of the surface of Cayley—it has ten tropes each of which meets the other nine in the base-lines of a range of plane curves of class 3. We shall frequently have occasion to consider the rational sextic curve (1), and hence (2) in its dual form. We shall show in this paper that (2) is a general symmetroid; that the reduction of the general symmetroid to the form (2) is possible in two ways; and hence, when the symmetroid is given, there are two forms (1) whose catalecticant gives the surface. Hence, it is seen that rational sextic curves in space are paired, either curve of a pair determining the other uniquely and in the same manner. Our main object is the discussion of the surface (2) in its relation to the rational sextic curve. Many features of the geometry on the surface are quite readily treated from this point of view. Cayley pointed out that the symmetroid may be birationally transformed into a quartic surface which is the Jacobian of a threefold linear system of quadrics. It may be proved that a cubic Cremona transformation is sufficient to accomplish this.

Certain special cases of the symmetroid are of considerable interest, for instance the Kummer surface, and the Hessian of a general cubic surface. The detailed consideration of these is reserved for a later memoir.

It is hoped that the beauty and suggestiveness of the hyperspatial methods used are made sufficiently apparent. While the arguments could perhaps be made briefer by direct analytical presentation, many geometrical relations between apparently unconnected theorems would be lost.

§ 2. *The Norm-Sextic in S_6, and the Rational Plane and Space Sextics.*

1. We shall use the letter R throughout this paper for the rational norm-curve in a space of six dimensions. This norm-curve is supposed to be non-degenerate, and chosen once for all. $S_0, S_1, \ldots, S_5$ will be used to denote flats lying in the given six-space, and of the dimension indicated by the subscript. S_0, S_1, S_2, S_3 are a point, a line, a plane, and a space, respectively. An S_p having $(p+1)$-point contact with R will be called "an S_p of R." S_k's meeting R once, twice, three times, ..., r times, will be called S_k's unisecant, bisecant, trisecant, ..., r-secant to R, respectively.

R may be given parametrically, in S_5's by

$$\xi_i = \binom{6}{6-i} t^{6-i} \qquad (i = 0, 1, \ldots, 6), \tag{3a}$$

and in points by

$$x_i = (-)^{6-i} t^i \qquad (i = 0, 1, \ldots, 6). \tag{3b}$$

An S_5, ξ, meets R in the six points whose parameters are the roots of

$$\xi_0 - \xi_1 t + \ldots + \xi_6 t^6 \equiv (\xi t)^6 = 0, \tag{4a}$$

and S_5's of R on a point x have contact with R at points whose parameters are given by

$$x_0 t^6 + 6 x_1 t^5 + 15 x_2 t^4 + \ldots + x_6 \equiv (x t)^6 = 0. \tag{4b}$$

(4a) and (4b) will be called "the sextic ξ" and "the sextic x," respectively, or merely "ξ" and "x" when there is no doubt as to the meaning. It follows from (4a) and (4b), and is, besides, known, that S_5's on a point x cut from R sextics apolar to the sextic x. S_{r-1}'s on x and r-secant to R meet R in the roots of the apolar r-ics of x. *In this fact lies the fundamental reason for the study of the plurisecant spaces of R to which we are about to proceed.* It will be seen for the rational sextic, and it is true in general, that the hyperspatial point of view unifies many apparently unconnected theorems relating to rational curves. As an example, there may be cited the intimate connection between the notion of osculants and that of perspective curves.*

Rational sextic curves in flats of dimension less than 6 may be derived from R by projection or section, these methods giving general rational curves of order 6 and class 6, respectively, if the given flats are in general position relative to R. We indicate the rational curve of order n in S_p by ρ_p^n, and that of class n in S_p by r_p^n. An invariant of (4a) or of (4b) equated to zero is the equation in S_6 of a contravariant or covariant five-way spread of R; similarly, the coefficients of a covariant of either of these forms equated simultaneously to zero give a spread covariantly connected with R. The operations of projection or section on these, by which a rational sextic is obtained from R, give covariant spreads of the resulting rational curve, whose relation to this curve is frequently easy to deduce.

2. If the sextic x have an apolar cubic, x is on a plane trisecant to R. By a simple enumeration, the locus of such points must be a five-way spread in S_6. The condition that x have an apolar cubic is the vanishing of its catalecticant. Hence,

(a) *The locus of planes trisecant to R has as equation the catalecticant of the sextic* (4b) *and is a five-way of order* 4.

We call this locus the spread G. Its equation is

$$|x| \equiv \begin{vmatrix} x_0, & x_1, & x_2, & x_3 \\ x_1, & x_2, & x_3, & x_4 \\ x_2, & x_3, & x_4, & x_5 \\ x_3, & x_4, & x_5, & x_6 \end{vmatrix} = 0. \tag{5}$$

Calling the three-way spread of bisecants to R, of order 10,† H, we now prove

(b) *The spread G contains H as a double spread.*

* Compare Conner: "Multiple Correspondences Associated with the Rational Plane Quintic Curve," *Transactions of the American Mathematical Society*, Vol. XIII, pp. 265–275.

† The rational plane sextic has 10 nodes.

The line joining two points x and y of S_6,

$$z_i = x_i + \lambda y_i,$$

meets (5) in four points whose parameters, λ_i, are the roots of

$$|z| = |x + \lambda y| = 0. \tag{6}$$

If $(x t)^6$ have an apolar quadratic, that is, if x is on a line bisecant to R, $(x t)^6$ may be reduced to the form

$$(x t)^6 = t^6 + \alpha$$

by a collineation on the parameter t, and λ^2 is a factor of (6), proving that the line xy meets (5) in two coincident points at x.

3. The surface (2) (p. 29) will be called the surface (of planes) S of ρ_3^6; dually, r_3^6 has a similarly defined contravariant surface (of points), Σ say. Cutting the S_5's of R by a space σ, we obtain a rational sextic of planes, r_3^6. G meets σ in the surface Σ of r_3^6. This surface may be defined as the locus of points of σ which determine, by planes of r_3^6 on them, catalectic sextics in the binary domain on r_3^6. Theorems (a) and (b) give

(c) *The surface Σ of r_3^6 is a ten-nodal quartic surface; the ten nodes are the points of intersection of σ with the three-way spread H. A node of Σ gives on r_3^6 a binary sextic reducible to the form $t^6 + \alpha$; that is, a cyclic sextic.*

These ten nodes will be called the cyclic points of r_3^6. Dually, ρ_3^6 has ten cyclic planes, the tropes of S.

4. A plane in S_6 has twelve constants. There are ∞^5 S_4's five-secant to R, and ∞^6 planes on each; thus it is a single condition on a plane in S_6 to carry an S_4 five-secant to R. This condition is easily calculated. If x, y and z are three points on such a plane, the sextics x, y and z have a common apolar quintic. Let $(a t)^5$ be this quintic; then

$$|x a|^5 (x t) \equiv 0,$$
$$|y a|^5 (y t) \equiv 0,$$
$$|z a|^5 (z t) \equiv 0,$$

for all t's. Eliminating the a's from these six equations, we obtain

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{vmatrix} = 0; \tag{7}$$

(7) is of degree 2 in the determinants $|x_i y_j z_k|$, the coördinates of the plane xyz. Hence,

(d) *Planes in S_6 carrying S_4's five-secant to R are in a quadratic system (hypercomplex) of ∞^{11} planes.*

From (d) we have that the locus of planes passing through a line, p, of S_6, and carrying S_4's five-secant to R, is a quadric spread. If $p = xy$, (7) is the equation of this spread in coördinates z. A quadric spread in S_6 containing ∞^1 S_4's must have a nodal plane. This plane in the present case passes through the line p, and is common to all S_4's on p and five-secant to R. We may see this in another way. There is a pencil of binary quintics apolar to all sextics of a given pencil, but a net of sextics apolar to a given pencil of quintics. Thus, giving p determines, by S_4's five-secant to R and on p, a pencil of quintics apolar to the pencil of sextics defined on R by points of p; to this pencil of quintics, however, is apolar a net of sextics determined by points of a plane through the line p. This plane is determined in a similar way by any of its lines; it belongs to a system of ∞^8 planes in S_6 — this system is useful in the study of the (unique) cubic surface passing through ρ_3^6 (§ 5).

5. In the space σ of r_3^6, we have from (d):

(e) *Planes whose points define on r_3^6 a linear system of binary sextics having a common apolar quintic are planes of a quadric surface, K, in σ.*

The surface K was discovered by Stahl,* and will be called the Stahl quadric of r_3^6. It is the locus of planes joining three at a time the sets of four stationary points (apparent cusps) of quartic osculants of the curve.

6. A pencil of binary sextics has in general a pencil of apolar quintics. But if the sextics are the first polars of a binary septimic, the pencil has a net of apolar quintics, namely, the apolar quintics of the septimic. Conversely, if a pencil of sextics have a net of apolar quintics, they are first polars of a septimic. Lines whose points give on R first polars of a binary septimic are in a system of ∞^7 lines in S_6; since the general line in S_6 has ten constants, it is three conditions on a line to belong to this system, and there is a ruled surface of these lines in the general space σ. Let the pencil of sextics

$$(x\,t)^6 + \lambda\,(y\,t)^6 = 0$$

have a net of apolar quintics; that is, let the line xy carry a double infinity of S_4's five-secant to R. Then to any third sextic $(z\,t)^6 = 0$ is apolar a unique quintic of the net. In geometric language,

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*Stahl, "Ueber die Fundamentalinvolutionen auf rationalen Curven," *Journal für Mathematik*, Vol. CIV, p. 56. In connection with this paper and with several others of Stahl, the author is inclined to think that many of his results were obtained by the use of hyperspace, and, for some reason—probably for greater convenience of statement—were subsequently translated into algebraic language. Compare the introductory pages of the above paper and those of the one entitled "Ueber die rationale ebene Curve vierter Ordnung," *Journal für Mathematik*, Vol. CI, p. 300. At all events, the results of Stahl are those most easily accessible to the hyperspace apparatus.

(f) *If a line xy of S_6 carry a double infinity of S_4's five-secant to R, any plane on xy carries a unique S_4 five-secant to R,*

with the immediate consequence,

(g) *The Stahl quadric K of r_3^6 is the locus of lines in σ which carry a double infinity of S_4's five-secant to R; i. e., it is the locus of lines whose points give on r_3^6 the first polars of a binary septimic.*

7. Closely connected with the subject of paragraph 5 is the system of lines in S_6 which carry S_3's four-secant to R. There are ∞^4 S_3's four-secant to R, and ∞^4 lines on each; there are thus ∞^8 lines in S_6 which carry S_3's four-secant to R, and there is hence a congruence of such lines in the general space σ. We desire to prove

(h) *Lines in a space σ and carrying S_3's four-secant to R are in a* (3, 6) *congruence;*

that is, three such lines pass through the general point, and six lie on the general plane of σ. S_3's on a point x of S_6 and four-secant to R meet R in roots of quartics apolar to x (∞^1 quartics) and lie on a four-way spread, this spread being the locus of points y such that the line xy carries an S_3 four-secant to R; *i. e.*, x and y have a common apolar quartic. Let $(kt)^4$ be this quartic. Then

$$|xk|^4 (xt)^2 \equiv 0,$$
$$|yk|^4 (yt)^2 \equiv 0;$$

eliminating k, we obtain

$$\left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 & x_5 & x_6 \\ y_0 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_2 & y_3 & y_4 & y_5 & y_6 \end{matrix} \right\| \equiv 0. \tag{8}$$

The order of this matrix in y is easily seen to be 3. Through a point x of σ there pass three lines of our congruence, since σ meets the four-way defined by (8) in three lines through x.

Again, let π be a general plane of S_6; projecting R from π, we obtain a rational space sextic ρ_3^6; let p be a fourfold secant line of ρ_3^6. The S_4 $p\pi$ meets R in four points, and the S_3 on these four points meets π in a line of the congruence in question. The lines of the congruence which lie on π are thus in (1, 1) correspondence with the fourfold secants of ρ_3^6, and there are hence six of these lines on π.*

* Pascal, "Repertorium der höheren Mathematik," II (Geometrie) (1902), p. 270.

8. Projecting R from a plane π, we obtain a space curve ρ_3^6. Plane sections of this curve are the projections of the sections of R by S_5's on π. It follows that points of π define on R, or on the section of R by π — an r_2^6 — binary sextics apolar to the plane-sections of ρ_3^6. ρ_3^6 and r_2^6 will be called *conjugate* curves. Points of π give on r_2^6 the fundamental involution of ρ_3^6; similarly, plane-sections of ρ_3^6 are the fundamental involution of r_2^6.

Dually, projecting from a space σ carrying a curve r_3^6 cut from R by σ, we obtain the conjugate sextic, ρ_2^6, of this curve. A node of ρ_2^6 arises when a line bisecant to R meets σ. These lines mark on σ the ten nodes of a symmetroid, the surface Σ of r_3^6. Hence,

(i) *The ten nodes of a rational plane sextic curve are in one-one correspondence with the ten nodes of a projectively definite symmetroid.*

§ 3. *The Stahl Quadric is a Rational Covariant Surface of the Symmetroid.*

9. There are ∞^2 six-planes circumscribing K and inscribed in the surface Σ of a sextic of planes, r_3^6.* Each of these six-planes is associated with one set of the fundamental involution of r_3^6. Regarding r_3^6 as cut from R by a space σ, an S_5 on σ meets R in a set of six points — a set of the fundamental involution of r_3^6 — and S_4's on five out of six these meet σ in one of the six-planes in question.† These six-planes are intimately connected with the osculants of r_3^6 and the perspective curves of the conjugate ρ_2^6. The author expects to publish an account of this theory later.

10. One of the above six-planes is uniquely determined when one of its twenty vertices (a point of Σ) is given. For, let x be such a vertex; there is a unique plane on x trisecant to R, and on this plane and σ a unique S_5 which determines a unique six-plane with vertex at x. Opposite vertices of these six-planes are hence in involutory and one-to-one correspondence, thus giving a birational transformation of Σ into itself which we shall call T. The principal theorem of this section is an immediate consequence of one of the properties of the fundamental curves of T.

A singular point of T can arise when, and only when, the plane on a point x of Σ and trisecant to R is indeterminate. This happens only at the nodes of Σ. Let $n_1, \ldots, n_{10}$ be the ten nodes of Σ. There is a line, p_1, on n_1 meeting R in two points, t_1 and t_2. Choosing a plane on t_1, t_2 and t_3, any third point of R, the S_5 on σ and the plane $t_1 t_2 t_3$ meets R in three further points, t_4, t_5, t_6. The point $t_4 t_5 t_6 \sigma$ (the trace on σ of the plane $t_4 t_5 t_6$) is on the fundamental

* Stahl, "Ueber Fundamentalinvolutionen," *loc. cit.*, pp. 56, 57.

† A complete verification of these statements would require the development of the theory of osculants of R. Cf. Conner, "Multiple Correspondences," etc., *loc. cit.*

curve corresponding by T to n_1. The locus of $t_4\,t_5\,t_6\,\sigma$ as t_3 varies is this fundamental curve, $S^{(1)}$ say. S_5's on σ and p_1 meet R in t_1, t_2 and in roots of quartics of a pencil. The points $t_3\,t_4\,t_5\,t_6$, mentioned above, are roots of such a quartic. The space σ meets the complete six-point $t_1 \ldots . t_6$ in six planes, of which four pass through n_1 and two meet on the line $t_3\,t_4\,t_5\,t_6\,\sigma$. Call this line p_λ, λ being the parameter of $t_3\,t_4\,t_5\,t_6$ in the pencil of quartics. On p_λ are four points of $S^{(1)}$, namely, $t_3\,t_4\,t_5\,\sigma$, $t_3\,t_4\,t_6\,\sigma, \ldots \ldots$ A pencil of binary quartics is apolar to a definite binary sextic, x, the pencil being cut from R by S_3's on x and four-secant to R. (x is here not on σ.) From the matrix (8), p. 34, we see that the locus of these S_3's is a cubic four-way. This enables us to find the order of the locus of p_λ as λ varies. A cubic four-way in S_6 can not meet a space in a surface of order greater than 2 without containing it entirely. It follows that the order of the locus of p_λ is not greater than 2. But an S_5 on σ may be made to contain the lines p_1 and p_2 on the nodes n_1 and n_2 of Σ and bisecant to R; hence the locus of p_λ is on all nodes of Σ but n_1. It follows that this locus is the quadric surface, Q_1, on the nodes $n_2, \ldots ., n_{10}$. There can not be a pencil of quadric surfaces on $n_2, \ldots ., n_{10}$, for in that case the pencil taken with Q_2 would give a net of quadrics on $n_3, \ldots ., n_{10}$, and no eight nodes of Σ are base-points of a net of quadrics.* Hence,

(j) *The nodes of Σ are singular points of the correspondence T. The fundamental curve corresponding to any node, n_1, is cut out of Σ by the quadric surface on the remaining nine nodes, and is a rational curve of order* 8 *with nine actual nodes.*

Recurring to our notation above, the point $t_4\,t_5\,t_6\,\sigma$ of $S^{(1)}$ may be named by the parameter t_3 of R. The plane $t_1\,t_2\,t_4\,t_5\,t_6\,\sigma$, on the point $t_4\,t_5\,t_6\,\sigma$, is a plane of the enveloping cone to the quadric K from n_1, and may be named, as a plane of this cone, by the same parameter, t_3. Projecting $S^{(1)}$ from n_1, we obtain a rational plane octavic curve, the enveloping cone to K from n_1 giving a perspective conic of this curve. *The enveloping cone to K from n_1 is thus unique when Σ is given,* since a rational plane octavic can have at most one perspective conic; two conics in one-one correspondence generate a rational quartic. Hence we obtain the principal theorem of this section:

(k) *Given the symmetroid Σ associated with a rational sextic curve, r_3^6, the enveloping cones to the Stahl quadric K from its ten nodes are uniquely determined. It follows that K is a rational covariant surface of Σ.*

§ 4. *The Two Sextics Determined by a Given Symmetroid.*

11. Let x be the symbol of a point in a space σ_x, and y that of a point in a second space σ_y. The equations

* Cayley, *loc. cit.*, and Pascal, "Repertorium," *loc. cit.*, p. 298.

$$\left.\begin{aligned} x_0 &= (\alpha y)^2, \\ x_1 &= (\beta y)^2, \\ x_3 &= (\gamma y)^2, \\ x_4 &= (\delta y)^2, \end{aligned}\right\} \tag{9}$$

α, β, γ and δ being four linearly independent quadric surfaces in σ_y, determine a one-to-eight correspondence, V,* between the spaces σ_x and σ_y. Planes of σ_x pass by V into quadric surfaces of a linear threefold system, C_y, in σ_y. Planes of σ_y, on the other hand, are sent by V into Steiner quartic surfaces in σ, a plane η of σ_y being mapped on its correspondent surface by means of the three-fold linear system of conics in which quadrics C_y meet η. To a point x correspond eight y's; the surface along which two y's with a common correspondent coincide is the Jacobian surface, J, of the system C_y. J is a quartic surface containing ten lines; there are ten quadrics C_y which degenerate into pairs of planes, and the line of intersection of two planes of such a pair is on J.

We use the notation $\xi\, p$, $\eta\, q$ for a plane and line of σ_x and σ_y, respectively. To a line p of σ_x corresponds the intersection of $V\xi$ and $V\xi'$, if ξ and ξ' are two planes on p; this curve is an elliptic quartic, the intersection of two quadrics C_y. VJ is a surface of order 16, since the quartic curve Vp meets J in sixteen points. VJ is of class 4, since there are four nodal quadrics in a pencil.

12. If in

$$(\xi x) = 0 \tag{10}$$

are substituted the values of x as given by (9), the discriminant as to y of (10) is the surface VJ in planes. If $(Ay)^2$ is the form (10), then VJ is

$$\begin{vmatrix} A_{00}, & A_{01}, & A_{02}, & A_{03} \\ A_{10}, & A_{11}, & A_{12}, & A_{13} \\ A_{20}, & A_{21}, & A_{22}, & A_{23} \\ A_{30}, & A_{31}, & A_{32}, & A_{33} \end{vmatrix} = 0, \tag{11}$$

* It is convenient, for the purposes of this paper, to discuss the correspondence V without reference to the norm-curve R, though it is of more than passing interest to point out how it arises from the notion of the norm-curve. The spread of S_4's of R is a five-way of order 10; it has a triple three-way spread of order 8, δ_8, any three S_4's of R meeting in a point of δ_8. Now it may be shown that δ_8 is mapped from a space σ_y by quadrics orthic to a norm-curve N in σ_y, and that N itself passes into R. Regarding the curve $\rho^6{}_3$ as obtained by projecting R from a plane π in S_6, the (8, 1) correspondence is here between the eight points in which a space on π meets δ_8 and the point which it determines on the space (σ_x) into which the projection is made. In particular, if π meets δ_8 six times, the quadrics $\alpha\,\beta\,\gamma\,\delta$ of the text become quadrics on six points, and the surface S of $\rho^6{}_3$ is a Kummer surface. Thus,

If a rational plane sextic $r^6{}_2$ has six triple points (dually, six triple tangents), the conjugate curve $\rho^6{}_3$ has a Kummer surface as covariant symmetroid.

The properties of the correspondence V which we use are so similar to those of the (1, 2) correspondence determined by (9) when $\alpha\,\beta\,\gamma\,\delta$ have six common points, that it does not seen necessary to develop them in great detail. See Sturm, "Die Lehre von den Geometrischen Verwandschaften," Vol. IV, part 12, and especially pp. 436 ff. Compare also Snyder, *Transactions of the American Mathematical Society*, Vol. XII, p. 354; Cayley, *loc. cit.*

a symmetroid (in planes) in σ_x. The surface S of a rational sextic curve

$$(a\xi)\,(\alpha t)^6 \equiv (m t)^6 = 0 \tag{1}$$

in σ_x is

$$\begin{vmatrix} m_0, & m_1, & m_2, & m_3 \\ m_1, & m_2, & m_3, & m_4 \\ m_2, & m_3, & m_4, & m_5 \\ m_3, & m_4, & m_5, & m_6 \end{vmatrix} = 0. \tag{2}$$

The form (11) involves twenty-four constants, and the rational space sextic curve has twenty-four constants. The question arises, Given the surface (11), is a reduction to the form (2) possible?

The equations

$$\left.\begin{aligned} y_0 &= t^3, \\ y_1 &= 3\,t^2, \\ y_2 &= 3\,t, \\ y_3 &= 1 \end{aligned}\right\} \tag{12}$$

represent a rational cubic curve in σ_y. The quadric orthic to the curve (12) (*i. e.*, apolar to the quadrics touching the planes of the curve) and cutting out the sextic $(a\,t)^6 = 0$ is

$$\begin{aligned} a_0 y_0^2 + a_2 y_1^2 + a_4 y_2^2 + a_6 y_3^2 + 2\,a_1 y_0 y_1 + 2\,a_2 y_0 y_2 + 2\,a_3 y_0 y_3 \\ + 2\,a_5 y_2 y_3 + 2\,a_4 y_3 y_1 + 2\,a_3 y_1 y_2 = 0, \end{aligned}$$

having the discriminant

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix} = 0;$$

thus it appears that if all quadrics C_y are orthic to the norm-curve (12), the equation of the symmetroid VJ, with properly chosen tetrahedron of reference in σ_y, assumes the form (2). The cubic curve (12) then maps by V into the rational sextic (1) whose catalecticant gives the surface (2).

Given a symmetroid of which (11) is the general form, the surface J is unique to within collineations, and the system C_y is unique, as is obvious from the form of (11). The problem of finding a rational sextic curve having a given symmetroid as surface S is thus reduced to that of finding a cubic curve orthic to every quadric of the system C_y. Reye* has shown that there are two such cubic curves, N_1 and N_2. N_1 and N_2 have the ten lines of J as common

* "Ueber lineare Systeme und Gewebe von Flächen zweiten Grades," *Journal für Mathematik*, Vol. LXXXII, pp. 78, 79.

axes. J is conversely determined by N_1 and N_2, as is easily shown.* We may state the preceding in the following theorem:

(l) *Given a symmetroid (of planes) in a space* σ_x, *there are two rational sextic curves (of points) having this surface as covariant surface* S. *The covariant surface* S *of a general rational sextic curve in space is a general symmetroid.*

Comparing theorem (g), § 2, and the results of § 3 with the above, we obtain

(m) *The two sextic curves which have a given symmetroid as covariant surface* S *have a common Stahl quadric* K. *The two sextics pair with the two systems of generators on* K *and also with the two systems of generators on a quadric* Q_i *on nine out of ten of the tropes of the given symmetroid.*

It follows that *if* K *has a node, every quadric* Q_i *is a conic.*

12. An interesting result may be drawn from the correspondence which V establishes between J and $VJ \equiv S$. $V\eta$ is a Steiner quartic surface, C_x, touching S along the correspondent of the quartic curve of intersection of J and η. Any line q maps by V into a conic touching S four times.

C_x is mapped from η by the intersections of η with the system C_y—a three-fold linear system of conics; that is, a system every conic of which is apolar to all conics of a range. Let the base-lines of this range be q_1, q_2, q_3, q_4. Since q_i^2 is apolar to all conics of the range, it follows that a quadric C_y, necessarily a cone, touches η along q_i; hence q_i is a line of a cone C_y, and the plane of Vq_i touches S at a point of Vq_i. The planes of Vq_i are the four tropes of C_x, while the points $V(q_i q_j)$ are the six pinch-points. It may readily be shown that J passes through the six points $q_i q_j$; in fact, J is the locus of points yy' apolar to all quadrics C_y, and $q_i q_j$, $q_l q_m$ are such a pair of points. The lines q_i map into the four conics in the tropes of C_x; these conics each touch S four times, three contacts of each being three pinch-points in a trope of C_x. It follows that the edges of the tetrahedron of tropes of C_x touch S at the pinch-points and

(n) *There are* ∞^3 *tetrahedra circumscribed to* S *whose edges touch* S.

These are analogous to the Humbert tetrahedra of the Kummer surface.†

§ 5. *The Cubic Surface on* ρ_3^6.

13. It is a single condition on a rational plane sextic curve to have a perspective conic. Stahl‡ showed that this condition is the condition for the

* Compare Meyer, "Apolarität und rationale Curven," § 31, p. 319.

† Hudson, "Kummer's Quartic Surface," p. 58.

‡ Stahl, "Zur Erzeugung der ebenen rationalen Curven," *Math. Ann.*, Vol. XXXVIII, pp. 565, 566.

existence of a binary octavic apolar to all line-sections of the curve. If $(a\,t)^6$, $(b\,t)^6$, $(c\,t)^6$ are three linearly independent line-sections, the condition is

$$\begin{vmatrix} a_0, & a_1, & a_2, & \dots, & a_6, & 0, & 0 \\ b_0, & b_1, & b_2, & \dots, & b_6, & 0, & 0 \\ c_0, & c_1, & c_2, & \dots, & c_6, & 0, & 0 \\ 0, & a_0, & a_1, & \dots, & a_5, & a_6, & 0 \\ 0, & b_0, & b_1, & \dots, & b_5, & b_6, & 0 \\ 0, & c_0, & c_1, & \dots, & c_5, & c_6, & 0 \\ 0, & 0, & a_0, & \dots, & a_4, & a_5, & a_6 \\ 0, & 0, & b_0, & \dots, & b_4, & b_5, & b_6 \\ 0, & 0, & c_0, & \dots, & c_4, & c_5, & c_6 \end{vmatrix} = 0. \qquad (13)$$

This is of degree 3 in the determinants

$$p_{ijk} = |\, a_i\, b_j\, c_k \,|$$

and also of degree 3 in the complementary determinants π_{lmnr} formed from four linearly independent sets of the fundamental involution of the given plane curve. Let (13) expressed in terms of the π's be

$$\phi\,(\pi_{lmnr}) = 0.$$

If now a, b, c are the forms generating the fundamental involution of a space sextic ρ_3^6, then any added form $(d\,t)^6$ defines planes on a point of the space of ρ_3^6, plane-sections on this point being apolar to a, b, c and also d. The determinants π_{lmnr} formed from a, b, c, d are proportional to determinants linear in the coördinates of the point.* Hence,

(o) *The locus of points from which ρ_3^6 projects into a plane sextic curve with a perspective conic is a cubic surface, P.*

The equation of this surface is intrinsically contained in (13).

A rational plane quintic curve has a unique perspective conic, and ρ_3^6 projects from one of its points into a rational quintic. Again, a plane sextic with a fourfold point has a perspective conic—the fourfold point repeated. Hence,

(p) *The cubic surface P is the unique cubic surface on ρ_3^6. It contains the fourfold secants of ρ_3^6.*

14. Certain facts regarding the surface P may easily be obtained by using the apparatus in S_6 which was developed in § 2 of this paper. We saw (theorem (d)) that it is a single condition on a plane in S_6 to carry an S_4 five-secant to R. Any two S_4's five-secant to R meet in a plane, and this plane carries ∞^1 S_4's five-secant to R. Let us call such a plane a plane $\bar{\omega}$. Since

* Stahl, "Zur Erzeugung der rationalen Raumcurven," *Math. Ann.*, Vol. XL, p. 2.

a plane $\bar{\omega}$ is uniquely determined by a pencil of binary quintics on R, there are ∞^8 planes $\bar{\omega}$ in S_6. We have shown that there is a unique plane $\bar{\omega}$ on every line of S_6, and that a plane $\bar{\omega}$ is equally well determined in this way by any line on it. If a plane $x\,y\,z$ carry ∞^1 S_4's five-secant to R, x, y and z have a pencil of apolar quintics, and they are second polars of an octavic, the pencil of quintics being the apolar quintics of the octavic. Any sextic apolar to x, y and z is apolar to the octavic. Hence,

(q) *R is projected from a space carrying a plane $\bar{\omega}$ into a sextic curve with a perspective conic.*

S_5's on the given space and on the S_4's on $\bar{\omega}$ and five-secant to R give the lines of the perspective conic. Again,

(r) *Spaces carrying planes $\bar{\omega}$ are in a cubic hypercomplex of spaces of which* (13) *is the equation, if a, b and c are three linearly independent S_5's.*

In a space carrying a plane $\bar{\omega}$, the Stahl quadric of r_3^6 is a conic in the plane $\bar{\omega}$. It follows that

(s) *The invariant conditions that a curve r_2^6 have a perspective conic and that the Stahl quadric of its conjugate ρ_3^6 have a node are identical.*

We need especially the following theorems:

(t) *Any plane on an S_3 four-secant to R is a plane $\bar{\omega}$.*

(u) *In an S_3 on a point of R there is a unique plane $\bar{\omega}$ on this point.*

Theorem (t) is sufficiently obvious. Theorem (u) follows, by projection from the point of R, from the fact that for the rational norm-curve R^5 in S_5 there is a unique line on a plane of S_5 carrying ∞^1 S_3's four-secant to R^5.*

16. Projecting R from the plane π on a space σ, we obtain a rational sextic curve, ρ_3^6 in σ; spaces on π carrying planes $\bar{\omega}$ meet σ in points of the cubic surface, P on ρ_3^6. On π there are six lines carrying S_3's four-secant to R (theorem (h)). Let us call these $p_1, \ldots, p_6$. The S_4 containing π and on an S_3 four-secant to R and meeting π in a line p_i, meets σ in a line q_i which lies entirely on P, since the pencil of planes on p_i and in the four-secant space are all planes $\bar{\omega}$. The lines q_i are the fourfold secants of ρ_3^6. In like manner, it follows from theorem (u) that P contains ρ_3^6, for on the space on π and a point of R there is a unique plane $\bar{\omega}$. This space marks on σ a point of ρ_3^6 and of P. The important additional point which we can make is that *there is a natural one-to-one correspondence between the lines of π and the points of P.* For,

* Marletta, "Sulle curve razionali del quinto ordine," *Rendiconti del Circolo matematico di Palermo*, Vol. XIX (1905), p. 94.

given a general line p on π, there is a unique plane $\bar{\omega}$ on it; on this plane and π there is an S_3 marking on σ a point of P. Conversely, given a point of P, the S_3 on this point and π carries a plane $\bar{\omega}$ meeting π in the line p. The six lines p_i are simple singular lines of the correspondence, for on a line p_i there is a pencil of planes $\bar{\omega}$. Hence,

(v) *The lines of π are mapped on the points of P by means of the linear threefold system of curves of class 3 on the six lines p_i. The correspondent of ρ_3^6 in π is a curve of class 10, Ψ say, having the lines p_i as fourfold tangents.*

On a plane of a space S_5 cutting out of a norm-curve R^5 a rational curve of class 5, r_2^5, there is a unique line carrying ∞^1 S_3's four-secant to R^5. This covariant line of r_2^5 was studied by the author* in a former paper; points of this line give on r_2^5 first polars of a binary sextic — the unique sextic apolar to the fundamental involution of r_2^5. Calling this covariant line of r_2^5 Ω, we have

(w) *The rational ten-ic Ψ is the locus of covariant lines Ω of quintic osculants of the curve r_2^6 in π.*†

The preceding theorems may be summed up in the following:

(x) *Given a plane sextic curve (of points), ρ_3^6, in a plane π, there are six points p_i of π having the property that lines on them cut from ρ_2^6 a pencil of sextics with a common apolar quartic. The locus of the covariant point Ω of quintic osculants of ρ_2^6 is a rational curve of order 10, Ψ, having p_i as fourfold points. Ψ is in one-to-one correspondence with ρ_2^6. A set of the fundamental involution of ρ_2^6 is cut from Ψ by a curve of order 3 on the six points p_i, the ∞^3 such curves giving the ∞^3 sets of the fundamental involution. Mapping π by means of this system of cubic curves gives a cubic surface P, and the ten-ic curve Ψ maps into the conjugate sextic of ρ_2^6. The lines p_i map into the fourfold secants of this conjugate curve; these fourfold secants are a sixer on P.*

The author suspects most strongly that the second sextic associated with the symmetroid of the conjugate curve lies also on the cubic surface P, and has as fourfold secants the other half of the double-six. This has not been proved.

BRYN MAWR COLLEGE, *December*, 1913.

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* Conner, *Transactions of the American Mathematical Society*, Vol. XII (1912), p. 265; see in particular, §§ 3, 6, 7, 8.

† See in particular § 6, Conner, *loc. cit.*